Lesson: Compact Metric Space

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1. <u>Learning outcomes</u>:

After studying this chapter students should be able to understand

- Whether a given set in a metric space is compact or not.
- Relationship between compactness and continuity.
- Relation between compactness and sequential compactness.
- Finite intersection property.
- Relation between compactness and Frechet Compactness.
- Relation between totally bounded sets and sequential compact sets.
- Lebesgue number.
- Application of Lebesgue Covering lemma.
- Heine – Borel theorem.

2. Introduction:

In this lesson we shall discuss the notion of compactness in a metric space. In first section we shall define compact set and we discuss certain theorems which characterize compact sets and give a complete description of compact sets in a metric space. Next we shall discuss the characteristics of a compact sets under continuous map, finite intersection property, and relations between compactness, sequential compactness and BW-compactness.

Some Definitions:

- **Cover**:
  Consider a nonempty set $\boldsymbol{X}$. Let $A \subseteq X$. a collection $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of subsets of X is said to be a cover of A if $A \subseteq \bigcup_{\alpha \in \Lambda} U\alpha$

- **Subover**:
  A sub-collection $\mathcal{U}_0$ of $\mathcal{U}$ is called a sub-cover of $\mathcal{U}$ for A if $\mathcal{U}_0$ is also a cover of **A.**

- **Open Cover:** In a metric space (X, d), a cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X is called an open cover of X if each U_α is open sets of (X, d).

3. Compact Metric Space

A metric space (X, d) is said to be compact if for each open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X $(\bigcup_{\alpha \in \Lambda} U\alpha = X)$, $\exists$ a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$ for X.

i.e. for each open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X, $\exists$ $\mathcal{U}_0 = \{U_{\alpha_i} : i = 1.2, \ldots, n\}$ such that $\bigcup_{i=1}^{n} Ui = X$.

- **Theorem 3.1:** Every finite set in a metric space is compact.

Proof: Let (X, d) be a metric space and $A = \{x_i : i = 1,2,..,n\}$ be a finite subset of X.

Let $\boldsymbol{\mathcal{U}} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of A.

Then $A \subseteq \bigcup_{\alpha \in \Lambda} U\boldsymbol{\alpha}$

So each $x_i \in U_{\alpha_i}$ for some U_{α_i} of the family $\boldsymbol{\mathcal{U}}$.

Therefore $A \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i})$

Hence $\{U_{\alpha_i} : i = 1.2, \dots, n\} = \mathcal{U}_0$ (say) is a finite subcover of $\boldsymbol{\mathcal{U}}$ for $\boldsymbol{A}$ and so $\boldsymbol{A}$ is compact■

- **Corollary:** Singleton set is compact.

- **Theorem 3.2 :** Every closed subset of a compact Metric Space is compact.

Proof: Let (X, d) be a compact metric space and A($\subseteq$ X) be closed. Let $\boldsymbol{\mathcal{U}} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of A in X.

Therefore $A \subseteq \bigcup_{\alpha \in \Lambda} U\alpha$.

Since A is closed, (X-A) is open in X.

Then $(\bigcup_{\alpha \in \Lambda} U\alpha) \cup (X - A)$ is clearly an open cover of X.

It is given that (X, d) is compact. Then that above cover of X has a finite sub-cover $\boldsymbol{\mathcal{U}}_0 = \{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}, (X - A)\}$ (say).

Therefore $(\bigcup_{i=1}^{n} U_{\alpha_i}) \cup (X - A) = X$

$\Rightarrow (\bigcup_{i=1}^{n} U_{\alpha_i}) \supseteq$ A

Let $\mathcal{F} = \{U_{\alpha_i} : i = 1,2,..,n\}$

Then $\mathcal{F}$ is a finite sub-cover of the open cover $\boldsymbol{\mathcal{U}}$ for A.

Hence A is compact■

- **<u>Theorem 3.3</u>:** every compact subset of a metric space is closed.

<u>Proof:</u> let (X, d) be a metric space and **A** be any compact subset of **X.**

To show that A is closed we will prove **A^c=(X-A)** is open in X.

For, let $\boldsymbol{y \in A^c}$ and $\boldsymbol{x \in A}$. then clearly x≠y ⇒ d(x,y)> 0

Let **d(x,y)=$\mathbf{r_x}$**, then the open sphere $S_{\frac{r_x}{2}}(x)$ *and* $S_{\frac{r_x}{2}}(y)$ are such that $S_{\frac{r_x}{2}}(x) \cap S_{\frac{r_x}{2}}(y) = \phi$

If $z \in S_{\frac{r_x}{2}}(x) \cap S_{\frac{r_x}{2}}(y)$, d(z,x)< $\frac{r_x}{2}$ and d(z,y)< $\frac{r_x}{2}$ and by triangle inequality

$d(x, y) \leq d(x, z) + d(z, y) < \frac{r_x}{2} + \frac{r_x}{2} = r_x$

Which contradicts the fact that **d(x,y)=$\mathbf{r_x}$.**

Now consider the collection $\boldsymbol{\mathcal{U}}$=$\{S_{\frac{r_x}{2}}(x) : x \in A\}$

Clearly $\bigcup_{x \in \boldsymbol{A}} S_{\frac{r_x}{2}}(x) \supseteq A$

Therefore $\boldsymbol{\mathcal{U}}$ is an open cover of $\boldsymbol{A}$.

Since A is compact set, ∃ a finite subcover $\boldsymbol{\mathcal{U}}_0$ of $\boldsymbol{\mathcal{U}}$ for $\boldsymbol{A}$.

Let $\boldsymbol{\mathcal{U}}_0$=$\{S_{\frac{r_{x_i}}{2}}(x_i) : i = 1,2,..,n\}$ so that $\bigcup_{i=1}^{n} S_{\frac{r_{x_i}}{2}}(x_i) \supseteq A$.

Let $B_y = \bigcap_{i=1}^{n} S_{\frac{r_{x_i}}{2}}(y)$.

Being finite intersection of open sets B_y is open containing ***y∈*A**^{c}.

Again for each $x_i \in A$, $\quad S_{\frac{r_{x_i}}{2}}(x_i) \cap S_{\frac{r_{x_i}}{2}}(y) = \phi$

$\Rightarrow \quad S_{\frac{r_{x_i}}{2}}(x_i) \cap B_y = \phi \quad \forall x_i$

$\Rightarrow \left(\bigcup_{i=1}^{n} S_{\frac{r_{x_i}}{2}}(x_i) \right) \cap B_y = \phi$

$\Rightarrow A \cap B_y = \phi$

$\Rightarrow B_y \subseteq A^c$

Since $y \in A^c$ is arbitrary, $\bigcup_{y \in A^c} B_y = A^c$

Since arbitrary union of open sets is open, A^c is open in X.

Hence A is closed■

- <u>Corollary</u> : A subset ***A*** of a compact metric space is compact if and only if ***A*** is closed.

- <u>**Theorem 3.4**</u>: every compact subset A of a metric space (X, d) is bounded.

<u>Proof:</u> Let $A(\subseteq X)$ be a compact .

Let us choose an open cover $\boldsymbol{\mathcal{U}}$ consisting of open spheres of unit radius.

i.e. $\boldsymbol{\mathcal{U}}=\{S_1(x): x \in A\}$.

Now $\bigcup_{x \in A} S_1(x) \supseteq A$

Since A is compact, $\boldsymbol{\mathcal{U}}$ has a finite sub-cover

$\boldsymbol{\mathcal{U}}_0=\{S_1(x_i): i = 1,2,..,n\}$.

Then $\bigcup_{i=1}^{n} S_1(x_i) \supseteq A$

$Let\ M = \max\{d(x_i, x_j): 1 \leq i \leq j \leq n\}$

Let $x, y \in A$ be any two elements, then $\exists$ elements x_i and x_j

Such that $x \in S_1(x_i)$ and $y \in S_1(x_j)$.

By triangle inequality

$d(x,y) \leq d(x,x_i) + d(x_i,x_j) + d(x_j,y) \leq 1 + M + 1 = M + 2$

$\Rightarrow A$ is bounded■

- **<u>Heine-Borel Theorem:</u>**

  Every closed and bounded subset of $\mathbb{R}$ is compact.

- **<u>Result:</u>** A subset S of $\mathbb{R}$ is compact if and only if it is closed and bounded.

❖ <u>4. Compactness and continuity</u>

In this section we will learn about the nature of a compact set under a continuous map.

- **<u>Theorem 4.1</u>**: Let (X, d_x) and (Y, d_y) be two metric spaces and $f: X \to Y$ be continuous. Then the continuous image of a compact subset A of X is compact in Y .

Proof: Let A be a compact subset of X and $\boldsymbol{\mathcal{U}}=\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $f(A)$.

Therefore $f(A) \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$

$\Rightarrow A \subseteq f^{-1}(\bigcup_{\alpha\in\Lambda} V_\alpha)$

$\Rightarrow A \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$

Since f is continuous and each V_α is open in Y, $f^{-1}(V_\alpha)$ are also open in X.

Hence $\mathfrak{F} = \{f^{-1}(V_\alpha): \alpha \in \Lambda\}$ is an open cover of A.

Since A is compact, ∃ a finite sub-cover

$\mathfrak{F}_0 = \{f^{-1}(V_{\alpha_i}): i = 1,2,..,n\}$.

Therefore $A \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$

$\Rightarrow A \subseteq f^{-1}(\bigcup_{i=1}^{n} V_{\alpha_i})$

$\Rightarrow f(A) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$

Hence $f(A)$ is compact■

- ✓ **Note:** The converse of the above theorem is not necessarily true. That is, if a function maps compact sets into compact sets, it does not always mean that the function is continuous.

Counter example:

Consider the Dirichlet function $f: \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1, if\ x \in \mathbb{Q} \\ 0, if\ x \notin \mathbb{Q} \end{cases}$$

Clearly f is not continuous map. But it maps every compact subset of $\mathbb{R}$ to a compact sub-set $\{0,1\}$(as it is finite subset) of $\mathbb{R}$(co-domain).

- **Corollary:**
  Let (X, d_x) *and* (Y, d_y) be two metric spaces and $f: X \to Y$ be continuous. If $A(\subseteq X)$ be a compact then $f(A)$ is closed and bounded in Y.
- **Corollary:**
  Let (X, d_x) *and* (Y, d_y) be two metric spaces and X *be compact*. *Let* $f: X \to Y$ be continuous. If $A(\subseteq X)$ be closed , then $f(A)$ is closed and bounded in Y.

❖ 5. Finite Intersection Property

A family $\mathcal{F}$ of subsets in a metric space (X, d) is said to have ***Finite Intersection Property*** (F.I.P) if every finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ has ***nonempty intersection***.

i.e. if $\mathcal{F}=\{F_\alpha: \alpha \in \Lambda\}$ be any family of subsets of X.

Then for every finite subfamily $\mathcal{F}_0=\{F_{\alpha_i}: i = 1,2,..,n\}$ $\bigcap_{i=1}^{n} F_{\alpha_i} \neq \phi$.

e.g. The family $\{[-n, n]: n \in \mathbb{N}\}$ of closed intervals of $\mathbb{R}$ has Finite Intersection Property.

➢ **Theorem 5.1:** A metric space (X, d) is compact If and only if for every collection of closed subsets $\mathcal{F}=\{F_\alpha: \alpha \in \Lambda\}$ in X having Finite Intersection Property, the intersection $\bigcap_{\alpha \in \Lambda} F_\alpha$ of the entire collection is nonempty.

Proof: Let (X, d) be compact metric space and $\boldsymbol{\mathcal{F}}$=$\{F_\alpha : \alpha \in \Lambda\}$ be any family of closed sets in (X, d) with F.I.P

If possible let $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$.

Now $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi \Rightarrow (\bigcap_{\alpha\in\Lambda} F_\alpha)^c = X$

$\therefore \bigcup_{\alpha\in\Lambda} F_\alpha^c = X$.

Since each F_α is closed, complement of each F_α is open in X.

i.e. $\{F_\alpha^c : \alpha \in \Lambda\}$ are family of open sets in X.

$\therefore \boldsymbol{\mathcal{U}}=\{F_\alpha^c : \alpha \in \Lambda\}$ is an open cover of X.

Since X is supposed to be compact ∃ a finite subcover $_0=\{F_{\alpha_i}^c$:i=1,2,..,n$\}$ of $\boldsymbol{\mathcal{U}}$ for X.

i.e. $\bigcup_{i=1}^{n} F_{\alpha_i}^c = X$

$\Rightarrow (\bigcup_{i=1}^{n} F_{\alpha_i}^c)^c = \phi$

$\Rightarrow \bigcap_{i=1}^{n} F_{\alpha_i} = \phi$

Now F_{α_i}'s are closed sets and $\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$ contradicts the fact that $\boldsymbol{\mathcal{F}}$ has F.I.P.

Hence $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

Conversely suppose every family of closed sets in (X, d) with F.I.P has nonempty intersection. We have to prove that X is compact.

It is quite similar to prove that for every family of closed sets in X with empty intersection does not have F.I.P. ⇒ X is compact.

For let $\boldsymbol{\mathcal{U}}=\{G_\alpha : \alpha \in \Lambda\}$be an open cover of X.

Then $\bigcup_{\alpha\in\Lambda} G_\alpha = X$ and taking complements we get

$$\bigcap_{\alpha \in \Lambda} G_\alpha^c = \phi$$

Since G_α's are open , G_α^c's are closed in X.

Then $\{G_\alpha^c : \alpha \in \Lambda\}$ is a family of closed sets in X whose intersection is empty.

Then by hypothesis this family does not have F.I.P. and so ∃ a finite subfamily say $\left\{G_{\alpha_i}^c : i = 1,2,..,n\right\}$ such that $\bigcap_{i=1}^n G_{\alpha_i}^c = \phi$

$\Rightarrow \left(\bigcap_{i=1}^n G_{\alpha_i}^c\right)^c = X$

$\Rightarrow \bigcup_{i=1}^n G_{\alpha_i} = X$

Hence $\left\{G_{\alpha_i} : i = 1,2,..,n\right\} = \mathcal{U}_0(say)$ is a finite subcover of the open cover $\boldsymbol{\mathcal{U}}$ for X.

Hence X

Is X compact■

- <u>**Definition**</u> **(relatively compact):** Let (X, d) be a metric space. A subset $\boldsymbol{A}$ is said to be **relatively compact** if $\bar{A}$ is compact in$\boldsymbol{X}$.

- <u>**Definition**</u>**(ε- net):** Let A be a subset of a metric space (X, d). Let $\varepsilon > 0$ be a real number. Then a non-empty subset B of A is said to be an $\varepsilon - net$ for set A if for any $a \in A$, ∃ a point $x \in B$ such that $a \in S_\varepsilon(x)$.

- <u>**Definition**</u>**(Totally Bounded Set/ Pre-compact):** A non-empty subset $\boldsymbol{A}$of a metric space (X, d) is said to be totally bounded if for any $\varepsilon > 0$ ∃ a **finite** $\varepsilon - net$ for A.

- **<u>Definition</u>(Lebesgue Number):** Let $\mathcal{U}= \{U_\alpha: \alpha\in \Lambda\}$ be an open cover of metric space (X, d). A real number $\lambda> 0$is called a Lebesgue Number for the open cover $\mathcal{U}= \{U_\alpha: \alpha\in \Lambda\}$ if for each subset A of X with $diam(A) < \lambda$, there is at least one U_α which contains A.

✓ **<u>Note</u>:** If λ is a Lebesgue number of an open cover, then any $\delta > \lambda$ is also a Lebesgue Number for that open cover.

➢ **<u>Theorem 5.2:</u>** In a metric space (X, d) , a subset A of X is compact implies it is totally bounded.

Proof: Let (X, d) be a metric space and $A(\subseteq X)$ be compact.

Then clearly for any $\varepsilon> 0$, $\mathcal{U} = \{S_\varepsilon(x): x \in A\}$ is an open cover of $\boldsymbol{A}$. By our hypothesis $A$is compact. Then $\exists$ a finite sub-cover say $\mathcal{U}_0 = \{S_\varepsilon(x_i): x_i \in A, i = 1,2,..,n\}$, then

$$A \subseteq \bigcup_{i=1}^{n} S_\varepsilon(x_i)$$

Let us consider the set $\{x_1, x_2, \dots, x_n\} = B\ (say)$. Then B is clearly a finite $\varepsilon - net$ for A. Hence A is totally bounded■

➢ **<u>Theorem 5.3</u>:** In a metric space (X, d) , if a subset A of X is totally bounded then it is bounded.

Proof: Let (X, d) be a metric space and $A(\subseteq X)$ be totally bounded. Then for any $\varepsilon > 0$ $\exists$ a **finite** $\varepsilon - net$ for A. Choose ε=1(>0) , $\exists$ finitely many points $x_1, x_2, .., x_n$ in A such that

$A \subseteq \bigcup_{i=1}^{n} S_1(x_i)$. Let $M = \max_{1\le i,j\le n}\{d(x_i, x_j)\}$. Then for any $x, y \in A$ with $x \neq$ y $\exists\, 1 \le i, j \le n$ $(i \neq$ j) such that $x \in S_1(x_i)$ and $\in S_1(x_j)$.

Thus from triangle inequality

$$d(x,y) \le d(x,x_i) + d(x_i,x_j) + d(x_j,y) < 1 + M + 1$$

$\Rightarrow d(x,y) < 2 + M$

This shows that $diam(A) < 2 + M$. Hence $\boldsymbol{A}$ is bounded■

- ✓ **Note:** The converse of the above theorem is not true in general. For example we discuss the following:

Consider the l_2 space consisting of real sequences $\{x_n\}$ such that $\sum_{i=1}^{\infty} x_i^2 < \infty$ and the metric is defined by $d(x,y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$, $x = \{x_n\}, y = \{y_n\} \in l_2$.

Further we consider the subset

$$A = \left\{x = \{x_n\} \in l_2 : \sum_{i=1}^{\infty} x_i^2 = 1\right\}$$

Or it can be defined as

$$A = \{x = \{x_n\} \in l_2 : d(x, \mathbf{0}) = 1\},$$

where **0=**$\{0,0,0,..,0\} \in l_2$.

For any two $x = \{x_n\}, y = \{y_n\} \in l_2$

$$d(x,y) \le d(x,\mathbf{0}) + d(\mathbf{0},y) = 1 + 1 = 2$$

This shows that A is <u>bounded</u>.

Consider the set $B = \{e_1, e_2, e_3, \dots \dots \dots\}$ of points of A, where $e_i = (0,0,0, \dots, 1_{i'th\ position}, 0,0, \dots \dots \dots \dots)$. Then for any $m, n \in \mathbb{N}$ $with\ m \neq n,\ d(e_m, e_n) = \sqrt{2}$. Observe that $\nexists$ any finite $\frac{1}{\sqrt{2}} -\ net$ for the set A.

To show, if possible let $\exists$ a finite $\frac{1}{\sqrt{2}} -\ net\ B = \{a_1, a_2, \dots, a_n\}$ for the set A. Then at least one $S_{\frac{1}{\sqrt{2}}}(a_k)$ contains infinitely many points of A.

Let $e_i, e_j \in S_{\frac{1}{\sqrt{2}}}(a_k)$. By triangle inequality

$$\sqrt{2} = d(e_m, e_n) \leq d(e_m, a_k) + d(a_k, e_n) < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

which is a contradiction. So there does not exists any finite $\frac{1}{\sqrt{2}} -\ net$ for the set A. Hence A is not totally bounded.

- ✓ **Note:** In a metric space any subset of a totally bounded set is totally bounded.

❖ 6. Sequential and Frechet compactness

Definition: A metric space (X, d) is said to be *sequentially compact* if every sequence $\{x_n\}$ of points of X has a convergent subsequence.

Definition: A metric space (X, d) is said to be *Frechet compact or BW –compact* if every infinite subset of X has a limit point in X

- ➢ **Theorem 6.1:** A metric space (X, d) is sequentially compact if and only if it is BW- compact.

Proof: Let (X, d) be sequentially compact. Also let A be any infinite subset of X. Consider a sequence $\{x_n\}$ of points of A. Since X is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to some $x \in X$.

i.e. $\lim_{k\to\infty} x_{n_k} = x$.

So for $\varepsilon > 0$ $\exists$ $k_0 \in \mathbb{N}$ for which the open sphere $S_\varepsilon(x)$ contains all x_{n_k} for all $k \geq k_0$.

i.e. $x_{n_k} \in S_\varepsilon(x)$ $\forall$ $k \geq k_0$.

Therefore x is a limit point of A.

Hence (X, d) is BW- compact.

Conversely, let (X, d) be BW-compact. Let $\{x_n\}$be an arbitrary sequence in $X$and suppose $A = \{x_n : n \in \mathbb{N}\}$. Then A is an infinite set in X.

If the range of the sequence is finite set, then a value is repeatedly occurs infinite times and then the sequence contains a constant subsequence which is obviously convergent. Hence (X, d) is sequentially compact.

If the range set of the sequence be infinite, then that set is an infinite set. Therefore it has a limit point in X say x (since (X, d) is BW-compact by hypothesis).

Since x is a limit point, for ε=1(>0) $S_1(x) \cap A$ is infinite. Choose an element $x_{n_1} \in S_1(x) \cap A$

For $n_2 > n_1$ *and* $\varepsilon = \frac{1}{2}$, choose $x_{n_2} \in S_{\frac{1}{2}}(x) \cap A$

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Proceeding in this way we get a subsequence $\{x_{n_k}: n_1 < n_2 < n_3, \ldots\ldots\}$ of $\{x_n\}$such that $d(x_{n_k}, x) < \frac{1}{k}$ for each $k \in \mathbb{N}$. This shows that the subsequence $\{x_{n_k}\}$ converges to x. Hence (X, d) is sequentially compact■

- **Lemma** (Lebesgue Covering Lemma):

Every open cover of a sequentially compact metric space (X, d)has a Lebesgue number.

Proof : Let $\mathcal{U}= \{U_\alpha: \alpha \in \Lambda\}$ be any open cover of X. Assume that it has no Lebesgue number. Then for each $n \in \mathbb{N}$, $\exists\, x_n \in X$ such that $S_{\frac{1}{n}}(x_n)$is not contained in any member U_α of $\mathcal{U}$. Since X is sequentially compact, the sequence $\{x_n\}$has a convergent subsequence$\{x_{n_k}: n_1 < n_2 < \cdots\}$ converges to some point $x \in X$.

i.e. $\lim_{n\to\infty} x_{n_k} = x$

Since $\mathcal{U}= \{U_\alpha: \alpha \in \Lambda\}$is an open cover of X, then $\exists$ one member U_β of $\mathcal{U}$ such that $x \in U_\beta$. As $U_\beta$is open, we can choose $\varepsilon > 0$ such that $U_\beta \supseteq S_{2\varepsilon}(x)$. Now $S_\varepsilon(x)$ contains all but finitely many terms of the subsequence $\{x_{n_k}\}$.In particular $\exists\, m \in \mathbb{N}\ with\ m > \frac{1}{\varepsilon}$ such that $x_m \in S_\varepsilon(x)$. Now let $y \in S_\varepsilon(x_m) \Rightarrow d(y, x_m) < \varepsilon \Rightarrow d(y, x) \leq d(y, x_m) + d(x_m, x) < \varepsilon + \varepsilon = 2\varepsilon$

$$\Rightarrow \text{y} \in S_{2\varepsilon}(x) \Rightarrow S_\varepsilon(x_m) \subseteq S_{2\varepsilon}(x).$$

Thus $S_{\frac{1}{m}}(x_m) \subseteq S_\varepsilon(x_m) \subseteq S_{2\varepsilon}(x) \subseteq U_\beta$. This contradicts the fact that $S_{\frac{1}{m}}(x_m)$ is not contained in any member of $\mathcal{U}$. Hence (X, d)has a Lebesgue number■

- **Theorem 6.2 :** A metric space(X, d) is compact $\Leftrightarrow$ (X, d) is sequentially compact.

Proof: Let (X, d) be compact and A is an infinite subset of X which has no limit point in X. So A is closed in X. Then for each $x \in A, there\ is\ an\ \varepsilon_x > 0$ such that $S_{\varepsilon_x}(x) \cap A = \{x\}$. Otherwise if there exists other points in $S_{\varepsilon_x}(x) \cap A$ other than x, x would be a limit point of A.

Clearly $\left(\bigcup_{x \in A} S_{\varepsilon_x}(x)\right) \cup (X - A)$ is an open cover of X which admits no finite subcover. This contradicts our hypothesis (X, d) is compact. Hence A must have a limit point in X. Since A is an arbitrary infinite subset of X , has a limit point in X implies (X, d) is BW-compact and hence by previous theorem it is sequentially compact.

Conversely , suppose that (X, d) be sequentially compact. Also let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Since (X, d) is sequentially compact therefore by Lebesgue Covering lemma

Has a Lebesgue number say $\delta > 0$. Also (X, d)being sequentially compact is totally bounded and so it has a finite $\frac{\delta}{3} - net$, say $\{x_1, x_2, \dots, x_n\}$.

Then $X = \bigcup_{i=1}^{n} S_{\frac{\delta}{3}}(x_i)$

Now for each $i, 1 \leq i \leq n$ we have $diam(S_{\frac{\delta}{3}}(x_i)) \leq \frac{2\delta}{3} < \delta$, so by definition of Lebesgue number there exists at least one U_{α_i} such that $S_{\frac{\delta}{3}}(x_i) \subseteq U_{\alpha_i}, \ \ i = 1,2,..,n$

$\Rightarrow \bigcup_{i=1}^{n} S_{\frac{\delta}{3}}(x_i) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

$\Rightarrow X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

Hence $\{U_{\alpha_i}: i = 1,2,..,n\}$ is a finite subcover of $\boldsymbol{\mathcal{U}} = \{U_\alpha : \alpha \in \Lambda\}$ for X and so (X, d) is compact■

- **<u>Theorem 6.3</u>**: A subset A of a metric space (X, d) is totally bounded if and only if every sequence in A has a Cauchy subsequence.

Proof: First let $A(\subseteq X)$be totally bounded and $\{x_n\}$ be a sequence in A. By total boundedness of A, $\exists$ a finite 1-net which covers A. Then at least one of these open balls must contain infinite number of terms of the sequence$\{x_n\}$. Let B_1be that open ball. Choose $x_{k_1} \in B_1$ for some $k_1 \in \mathbb{N}$. Being a subset of $\boldsymbol{A}$ which is totally bounded set, $B_1$is also totally bounded. Hence B_1 can also be covered by a finite $\frac{1}{2} - net$, at least one of which contains infinite number of terms of the sequence $\{x_n\}$. Let B_2 be that open ball. Choose $k_2 \in \mathbb{N}$ such that $k_2 > k_1$ and $x_{k_2} \in B_2$. Proceeding in this way we have for each $n \in \mathbb{N}$ open balls $B_n(\subseteq B_{n-1} \subseteq B_{n-2} \subseteq \cdots \subseteq B_2 \subseteq B_1)$ if radius $\frac{1}{n}$ such that $x_{k_n} \in B_n \ \ with \ \ k_1 < k_2 < \cdots < k_n$. Clearly $\{x_{k_n}\}$is a Cauchy subsequence of the sequence $\{x_n\}$

In fact $\forall\, i,j \geq n$, $x_{k_i}, x_{k_j} \in B_n$ so that $d\left(x_{k_i}, x_{k_j}\right) < \frac{2}{n} < \varepsilon$ if we choose $n > \frac{2}{\varepsilon}$.

Conversely, let A be a subset of a metric space $(X,d)$and every sequence in Ahas a Cauchy subsequence. We show that A is totally bounded.

If possible let $\boldsymbol{A}$ is not totally bounded. Then for some $\varepsilon > 0$, A has no finite $\varepsilon - net$. If $x_1 \in A$, there must be some $x_2 \in A$ such that $d(x_1, x_2) \geq \varepsilon$. Otherwise $\{x_1\}$ would be a finite $\varepsilon - net$ in A. Similarly since $\{x_1, x_2\}$ cannot be an $\varepsilon - net$ in $\boldsymbol{A}$, $\exists\, x_3 \in \boldsymbol{A}$ such that $d(x_1, x_3) \geq \varepsilon$ and $d(x_3, x_2) \geq \varepsilon$. Proceeding in this way we constrict a sequence $\{x_n\}$ of points of A such that for $m \neq n$ $d(x_m, x_n) \geq \varepsilon$.

It is clear that this sequence cannot have a Cauchy subsequence. This is a contradiction to our assumption. Hence $\boldsymbol{A}$ must be totally bounded■

- **Prove that a metric space $(\boldsymbol{X}, \boldsymbol{d})$ is sequentially compact if and only if it is complete and totally bounded.**

<u>**Proof:**</u> Let (X,d) be sequentially compact. Then every sequence $\{x_n\}$ of points of X has a convergent subsequence in X . Let $\{x_{n_k}\}$ be the convergent subsequence of $\{x_n\}$ which converges to the point $x \in X$. Since every convergent sequence in a metric space is a Cauchy sequence, $\{x_{n_k}\}$ is a Cauchy subsequence of $\{x_n\}$. Hence every sequence in (X,d) have a Cauchy subsequence.Then by *<u>Theorem 6.3</u>* (X,d) is totally bounded.

Now we show that $(X, d)$is complete. Let $\{x_n\}$ be a Cauchy sequence in X. Then for sequentially compactness $\{x_n\}$ has a convergent subsequence. By the result " A Cauchy sequence in a metric space (X, d) is convergent if and only if it has a convergent subsequence" the given Cauchy sequence converges. This proves that $(X, d)$is complete.

Conversely, Let (X, d) be complete and totally bounded and $\{x_n\}$ be a sequence in (X, d). Then by *Theorem 6.3* the sequence $\{x_n\}$has a Cauchy subsequence $\{x_{n_k}: n_1 < n_2 < \cdots.\}$. As $(X, d)$is complete, $\{x_{n_k}\}$converges in X. Hence (X, d) is totally bounded■

7. References:

1. S.C Malik, Savita Arora, Mathematical Analysis (Second Edition)
2. Manabendra Nath Mukherjee, Elements of Metric Spaces (Second Edition,2008)